Report of activities SCAT project

Cesar Gomez

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Participation on seminars

Méthode de Monte-Carlo pour les EDP non linaires.

Nizar Touzi, (CMAP, Ecole Polytechnique).

23 September, 2007.
The main major of this seminar was presenting a new generalization of the well known *Feynman-Kac's* formula for linear diffusions to the case of fully nonlinear PDEs, i.e., trying to arrive to a stochastic representation of the last ones and eventually try also Monte-Carlo methods to their numerical approximation.
Let us consider standard Brownian motion W_t on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with the filtration generated by W itself. If there exists an $\mathcal{F}_t^W$-adapted process (Y, Z) such that

$$\begin{aligned} Y_t =& \xi + \int_t^T F_r(Y_t, Z_t)dr - \int_t^T Z_r \cdot dW_r \\ \text{i.e.} \quad & dY_t = -F_t(Y_t, Z_t)dt - Z_t dW_t, \text{and} \quad Y_T = \xi(\omega) \end{aligned}$$

where $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$, and $\{F_t(y, z), t \in [0, T]\}$ is $\mathcal{F}^W$-adapted, then we say that (Y, Z) is a solution of the *backward stochastic differential equation* (BSDE)(**??**). For example if F is Lipschitz in (y, z) uniformly in (ω, t), and $\xi \in \mathbb{L}^2(\mathbb{P})$, then there is a unique solution satisfying

$$\mathbb{E}[\sup_{t \leq T} |Y_t|^2] + \mathbb{E} \int_0^T |Z_r|^2 dr < \infty.$$

Now let us concentrate ourselves in Markovian BSDEs, let us consider X defined by the (forward) SDE

$$\begin{aligned} & dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ \text{and} \qquad & F_t(y, z) = f(t, X_t, y, z), \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ & \xi = g(X_T) \in L^2(\mathbb{P}), \quad g : \mathbb{R}^d \to \mathbb{R}. \end{aligned}$$

If f is continuous, Lipschitz in (x, y, z) uniformly in t, then there is a unique solution to the BSDE

$$dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t \cdot dW_t, \qquad Y_T = g(X_T).$$

More over there exists a measurable function V such that $Y_t = V(t, X_t), \qquad 0 \leq t \leq T$. So by definition we have that

$$\begin{aligned} Y_s - Y_t &= V(s, X_s) - V(t, X_t) \\ &= -\int_t^s f(X_r, Y_r, Z_r)dr + \int_t^s Z_r \cdot dW_r \end{aligned}$$

Next if V is smooth we have by *Itô' s* lemma that

$$\begin{aligned} \int_t^s \mathcal{L}V(r, X_r)dr + & \int_t^s DV(r, X_r) \cdot dW_r \\ & = -\int_t^s f(X_r, Y_r, Z_r)dr + \int_t^s Z_r \cdot \sigma(X_r)dW_r. \end{aligned}$$

Here $\mathcal{L}$ is the Dynkin operator associated to X:

$$\mathcal{L}V = V_t + b \cdot DV + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V].$$

So under some conditions, the semilinear PDE

$$\begin{aligned} & -\frac{\partial V}{\partial t} - \mathcal{L}V(t, x) - f(x, V(t, x), DV(t, x)) = 0 \\ & V(T, x) = g(x). \end{aligned}$$

Has a unique solution which can be represented as $V(t, x) = Y_t^{t,x}$, where $Y^{t,x}$ solves the following BSDE,

$$\begin{aligned} & Y_s = g(X_T) + \int_s^T f(X_r, Y_r, Z_r)dr - \int_s^T Z_r \cdot \sigma(X_r)dW_r, \qquad t \leq s \leq T, \\ \text{and} \qquad & X_t = x, dX_s = b(X_s)ds + \sigma(X_s)dW_s, \qquad t \leq s \leq T. \end{aligned}$$

Them finally the author of this work make an interesting generalization of these facts to derive stochastic representations of fully nonlinear PDEs of the form (in particular, representation of general stochastic control problems).

$$-v_t(t,x)+f(t,x,v(t,x),Dv(t,x),D^2v(t,x))=0 \qquad (t,x)\in[0,T)\times\mathbb{R}^d \quad (1)$$

with terminal condition

$$v(T,x)=g(x).$$

Their results in short establish under certain regularity assumptions that if the solution of (1) is regular enough, then a the same time the processes

$$\begin{aligned}
Y_t &= v(t,X_t^{s,x}), \qquad t\in[s,T],\\
Z_t &= Dv(t,X_t^{s,x}), \qquad t\in[s,T],\\
\Gamma_t &= D^2v(t,X_t^{s,x}), \qquad t\in[s,T],\\
A_t &= \mathcal{L}Dv(t,X_t^{s,x}), \qquad t\in[s,T].
\end{aligned}$$

Solve the following BSDE of *second order*

$$\begin{aligned}
dY_t &= f(t,X_t^{s,x},Y_t,Z_t,\Gamma_t)dt+Z_t\circ dX_t^{s,x}, \qquad t\in[s,T),\\
dZ_t &= A_t dt+\Gamma_t dX_t^{s,x}, \qquad t\in[s,T),\\
Y_T &= g(X_T^{s,x}).
\end{aligned}$$

Here $Z_t\circ dX_t^{s,x}$ denotes *Fisk-Stratonovich* integration, which is related to Itô integration by

$$Z_t\circ dX_t^{s,x}=Z_t dX_t^{s,x}+\frac{1}{2}\mathrm{Tr}[\Gamma_t\sigma(X_t^{s,x})\sigma^T(X_t^{s,x})]dt.$$

And $\mathcal{L}$ is the infinitesimal generator of the diffusion $X_t^{s,x}$ given by

$$\begin{aligned}
&dX_t=\mu(X_t)dt+\sigma(X_t)dW_t,\\
&X_s=x.
\end{aligned}$$

Once having the stochastic representation to PDEs of the form (1) a very interesting question that comes immediate to mind is about the possibility of applying Monte-Carlo methods to approximate the solution.

Market Completion Using Options.

JAN OBLOJ, (Imperial College London).

Nov 2007.
It is well known that the Black-Scholes financial market model, consisting of a price diffusion and a non-random money market account, is complete, i.e., every contingent claim is replicated by a portfolio formed by dynamic trading in the two assets. As soon as oneself consider more realistic models in order to correct the empirical deficiencies of the B-S model, for example one might consider models that include stochastic volatility or jumps in the asset process, then completeness is lost if we continue to regard the original two assets as the only tradable. In this seminar some criteria are exposed under which trading in the underlying and a finite number of options completes the market in a model that takes into account stochastic volatility. Here is a brief description of this talk.

Consider a market in which an investor can trade in d assets $A = (A^1, A^2, \ldots, A^d)$. It is assume that there is not arbitrage in the market. Market *completeness* is investigated in $[0, T]$. We therefore assume the existence of an equivalent martingale measure $\mathbb{P}$ under which trading in this market is fear, we also consider that the market factors are modeled with a d-dimensional diffusion process $(\xi_t)_t \geq 0$, solution of the following SDE:

$$d\xi_t = m(t, \xi_t)dt + \sigma(t, \xi_t)dW_t, \qquad \xi_0 = x_0 \in \mathcal{D}. \tag{2}$$

Here W_t is d-dimensional Brownian motion in a probability triplet $(\Omega, \mathcal{L}, \mathbb{P})$ w.r.t. its natural filtration, and $\mathcal{D}$ is an open connected set. We make the following assumption:

$$\begin{aligned}&(2) \text{ has a unique strong solution with } \quad \mathbb{P}(\xi_t \in \mathcal{D}) = 1, t \geq 0,\\ &\sigma(t,x)\sigma(t,x)^T \quad \text{is extrictly positive for a.e.} \quad (t,x) \in [0, T \times \mathcal{D}].\end{aligned}$$

The traded assets are of European type, asset A^i has a given payoff $h_i(\xi_{T_i})$ at maturity T_i, larger than the time-horizon on which we investigate completeness $T \leq T_i$. As we work under the risk neutral measure $\mathbb{P}$, the discounted price process of an asset is a martingale, i.e.,

$$A^i = \mathbb{E}\big[e^{-r(T_i - t)} h_i(\xi_{T_i}) \big| \mathcal{F}_t\big], \quad 0 \leq t \leq T_i. \tag{3}$$

We consider next the semi-group of (ξ_t) denoted by $(P_{u,t})$, i.e. $P_{u,t}h(x) = \mathbb{E}^{x,u}[h(\xi_t)] := \mathbb{E}[h(\xi_t)|\xi_u = x], \quad u \leq t$. The Markov property of (ξ_t) implies that

$$A^i = v_i(t,\xi), \quad \text{where} \quad v_i(t,\xi) = e^{-r(T_i-t)} P_{t,T_i} h_i(x).$$

We assume that v_i are of class $\mathcal{C}^{1,2}$ on $(0,T) \times \mathcal{D}, \quad 1 \leq i \leq d$.
Let $G(t,x)$ be the matrix of partial derivatives,

$$G(t,x) = \big(\frac{\partial v_i(t,x)}{\partial x_j}\big)_{1 \leq i,j \leq d}$$

The following *practical* criteria for market completeness is t he main target of t his seminar.

Theorem 1 *Under the assumptions done above and if further $m_i, \sigma_i : (0,T) \times \mathcal{D} \to \mathbb{R}, \quad 1 \geq i \geq d$, are real analytic functions. Then the market is complete if and only if there exists a point $(t_0, x_0) \in (0,T) \times \mathcal{D}$ such that $G(t_0, x_0)$ and $\sigma(t_0,x_0)\sigma(t_0,x_0)^T$ are non singular.*

Hedging and Optimization in a Geometric Additive Market.

JOSÉ M. CORCUERA, (University of Barcelona.).

30 November, 2007.
The exposition occurs in 5 steeps as follows:

- An additive model of asset prices is proposed.
- The existence and nature of equivalent martingale measures are investigate.
- The possibility of completing the market including some other assets tradable(i.e. for example options) in considered.
- Then the hedging of a portfolio is analyzed.
- Last, the problem of portfolio optimization is also considered.

Let us present the main ideas. The market model proposed consists of an exponential additive model formed by a risk free bond $B = \{B_t, t \geq 0\}$, where $B_t = \exp(\int_0^t r_s ds)$, with r_s deterministic and a risky stock $S = \{S_t, t \geq 0\}$ that verifies:

$$\frac{dS_t}{S_{t^-}} = dZ_t, \qquad S_0 > 0. \tag{4}$$

Here Z is a natural additive process with local characteristics (with respect to the Lebesgue measure) (c_t^2, ν_t, γ_t), that means in more familiar terms using the Lévi-Itô decomposition that our process Z can be written as

$$Z_t = \int_0^t c_s dW_s + X_t. \tag{5}$$

Here W_t is standard Brownian motion and $X = \{X_t, t \in [0,T]\}$ is a jump process independent of W, this jump part is given by

$$\begin{aligned} X_t = &\int_{\{s\in(0,T],|x|<1\}} x(J(ds,dx) - \nu_s(dx)ds) \\ &+ \int_{\{s\in(0,T],|x|\geq 1\}} xJ(ds,dx) + \int_0^t \gamma_s ds. \end{aligned}$$

Here $J(ds,dx)$ is a Poisson random measure on $[0,T]\times\mathbb{R}-\{0\}$ with intensity measure $\nu_t(dx)dt$. The next theorem talks about the relation between two equivalent measures in this model.

Theorem 2 *Let $Z = \{Z_t, 0 \leq t \leq T\}$ be an additive process with local characteristics $(c_t^2, \nu_t, \gamma t)$. Then if there is a probability measure Q equivalent to P, such that Z is a Q-natural additive process with local characteristics (with respect to the Lebesgue measure) $(\bar{c}_t^2, \bar{\nu}_t, \bar{\gamma}_t)$ we have*

1. *$\bar{\nu}_t(dx) = H(t,x)\nu_t(dx)$ for some Borel function $H(t,x)\mathbb{R}^+ \times \mathbb{B} \to (0,\infty)$.*
2. *$\bar{\gamma}_t = \gamma_t + \int_{-\infty}^{+\infty} x\mathbf{1}_{\{|x|<1\}}(H(t,x)-1)\nu_t(dx) + G_t c_t^2$ for some Borel function $G : \mathbb{R}^+ \to (0,\infty)$.*
3. *$\bar{c}_t = c_t$. For every $0 \leq t \leq T$.*

It happens that the solution of (4) looks like

$$S_t = S_0 \exp\big(Z_t - \frac{1}{2}\int_0^t c_s^2 ds\big) \prod_{0<s\leq t} (1+\Delta Z_s)\exp(-\Delta Z_s). \tag{6}$$

Here $\Delta Z_s = Z_s - Z_{s^-}$ denotes the jump of Z at time s. Then from theorem 2 if we consider S_t in (6), and write it in the dynamics corresponding to an equivalent probability measure with the local characteristics $(\bar{c}_t^2, n\bar{u}_t, \bar{\gamma}_t)$ we will obtain

$$\begin{aligned} \bar{S}_t = S_0 \exp\big(&\int_0^t c_s d\bar{W}_s + \bar{L}_t + \int_0^t \big(a_s - r_s + G_s c_s^2 - \frac{c_s^2}{2}\big)ds\big) \\ \times \exp\big(&\int_0^t \int_{-\infty}^{+\infty} x(H(t,x)-1)\nu_s(dx)\big)ds \prod_{0<s\leq t}(1+\Delta\bar{L}_s)\exp(-\Delta\bar{L}_s). \end{aligned} \tag{7}$$

$\bar{W}_t = W_t - \int_0^t G_s c_s ds$ is Q-Brownian motion, also the process X_t has the Q-Doob-Meyer decomposition

$$X_t = \bar{L}_t + \int_0^t a_s ds + \int_0^t \int_{-\infty}^{+\infty} x(H(s,x)-1)\nu(dx)ds.$$

Here $\bar{L}_t$,is a Q-martingale and $\bar{\nu}_t(dx) = H(t,x)\nu_t(dx)$, for all $0 \leq t \leq T$. Next the necessary and sufficient condition for $\bar{S}_t$ to be a Q-martingale rests

$$G_t c_t^2 + a_t - r_t + \int_{-\infty}^{+\infty} x(H(t,x)-1)\nu_t(x) = 0.$$

Next one completes the market considering the following martingales

$$H_t^{(i)} = Z_t^{(i)} - \mathbb{E}_Q(Z_t^i).$$

Here $Z_t^{(i)} = \sum_{0<s\leq t}(\Delta Z_s)^i, \quad i \geq 2$, in such way that if (M_t) is any Q-squared-integrable contingent claim X. En particular for the martingale $M_t := \mathbb{E}_Q[(e^{-\int_0^T r_s ds})X|\mathcal{F}_t]$ Then

$$dM_t = \sum_{k=1}^{\infty} \beta_t^k d\bar{H}_t^{(k)}. \tag{8}$$

Here $\bar{H}^{(k)}$ are the orthogonal version of the processes $H_t^{(k)}$, the β_t^k are predesible processes that are interpreted as the strategy of replication for the contingent claim X. Finally we recall that a result of representation like that on (8) allowes one then to develop the classical ideas and calculations used to solve the hedging and the optimal portfolio allocation problems.

Join conditional density of a Markov process and its local time with applications to default risk modeling.

UMUT ÇETIN, (Department of statistics, London School of Economics).

23 November, 2007.
Suppose that the default time of a certain firm is modeled by some τ which is a positive random variable defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$.

A standard assumption is that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > 0) > 0$ for all $t \in \mathbb{R}_+$. There is also a reference filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ that models the flow of information obtained from relevant (or irrelevant) asset prices, news accounting information, etc. Typically τ is not an $\mathcal{F}$-stopping time, that is the event $\{\tau > t\}$ can not be decide from $\mathcal{F}_t$, but since default is a public event, a new filtration $\mathcal{G}$ which is obtained by enlarging $\mathcal{F}$ just enough so that τ becomes a $\mathcal{G}$-stopping time is considered. The interest is then over probabilities like $\mathbb{P}(\tau > T|\mathcal{G}_t)$, which would give a price for the defautable zero-coupon bond provided $\mathbb{P}$ is some risk-neutral measure. The key formula that links the two filtrations $\mathcal{F}$ and $\mathcal{G}$, due to **Dellacherie** is that for any $Y \in \mathcal{H}$

$$\mathbb{E}[\mathbf{1}_{[r>t]}Y|\mathcal{G}_t] = \mathbf{1}_{[r>s]}\frac{\mathbb{E}[\mathbf{1}_{[r>t]}Y|\mathcal{F}_t]}{\mathbb{P}(r > s|\mathcal{F}_t)}, \qquad \text{for} \quad s \leq t.$$

Applications to default risk

Consider a company which issues a bond with face value of \$1 and maturity $T > 0$. Let θ_t a proxy for the firm value such that

$$d\theta_t = \sigma(\theta_t)dW_t + \mu(\theta_t)dt, \tag{9}$$
$$\theta_0 = 0. \tag{10}$$

Here $\sigma : \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$ are Lipschitz. suppose moreover that $\sigma(\cdot) > 0$. Let τ the first time that θ falls below $a < 0$. The firm will default and wonť make payments if $\tau \leq T$. that is why, the set of probabilities $\mathbb{P}(\tau > t|\mathcal{F}_t)$ are important in this problem, to analyze them we start by the most simple case of studding the situation weather he Brownian motion reach a certain level.

Definition 1 *Let W standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Define for $a < 0$,*

$$\tau_a := \inf\{t > 0 : W_t = a\}.$$

*Let L^x be the **local time** process of W at level $x \in \mathbb{R}$ which could be defined by the following a.s. limit*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{[x \leq W_s \leq x+\epsilon]} ds. \tag{11}$$

Among other properties of the local time process we have

- It satisfies for each $t \geq 0$,

$$|W_t - x| - |x| = \int_{0^+}^t \operatorname{sgn}(W_s - x)dW_s + L_t^x.$$

- L_t^x is continuous and increasing for each x.

- (*Occupation times formula*) For any bounded measurable real function g,

$$\int_{-\infty}^{\infty} L_t^x g(x)dx = \int_0^t g(W_s)ds.$$

- $L_t^0 > 0$ for all $t > 0$.

It is also known that in this normal(Gaussian) case the distribution of the local time is,

$$\mathbb{P}(L_t^x \leq y) = 2\Phi\big(\frac{y + |x|}{\sqrt{t}}\big) - 1.$$

Here Φ is the cumulative distribution function of standard normal. Now recall $a < 0$, we have then that,

$$\begin{aligned}
\mathbb{P}(L_t^a = 0) &= 2\Phi\big(- \frac{a}{\sqrt{t}}\big) - 1 \\
&= 1 - 2\Phi\big(\frac{a}{\sqrt{t}}\big) \\
&= 1 - 2\mathbb{P}(W_t \leq a) = \mathbb{P}(\tau_a > t).
\end{aligned}$$

due to the reflexion principle of Brownian motion. But let us remember that we are interested in more general cases than that of Brownian motion. Going back to our model of default risk (9), we remark that the process θ_t is no publicly observable but one extracts information about it from the market trough a process Y which satisfies,

$$\begin{aligned}
dY_t &= dB_t + \alpha(t, \theta_t, Y_t)dt, \\
Y_0 &= 0.
\end{aligned}$$

Here α is Lipschitz, and B_t is a Brownian motion whose quadratic variation with respect to W_t is given by,

$$\frac{d}{dt}[B, W]_t = \rho(t, \theta_t, Y_t).$$

Suppose $\tau := \inf\{t > 0 : \theta_t = a\}$, for $a < 0$ is finite a.s. so we are not dealing with a vacuous problem. We remark that we are interested in probabilities that concern the distribution of τ. We recall that it models a default situation. To this end are considered some generalizations of the calculations that are done in the case of Brownian motion, in this last case for example some processes related with τ are considered in order to obtain information about the distribution of τ, by such information we refer ourselves to the densities of such processes for example. this densities are obtain more or leas as follows. Consider the family of processes,

$$X_t = \frac{1}{\epsilon}\int_{0+}^{t} \mathbf{1}_{[a\leq\theta_s\leq a+\epsilon]}\sigma^2(s,\theta_s)ds.$$

It happens that X_t^ϵ converges to L_t^a a.s. for every t as $\epsilon \to 0^+$. L^a is the local time of θ at a. Let us introduce the following notation,

$$\begin{aligned}
g_t^L(x) &:= \mathbb{P}[L_t^a \in dx|\mathcal{F}_t^Y]/dx, \qquad \text{for} \quad x \in \mathbb{R}^+,\\
g_t^{L\theta}(x,\theta) &:= \mathbb{P}[L_t^a \in dx, \theta_t \in d\theta|\mathcal{F}_t^Y]/dxd\theta, \qquad \text{for} \quad (x,\theta) \in \mathbb{R}^+ \times \mathbb{R}\\
\mathcal{L}^* g_t^{L\theta}(x,\theta) &= -\delta_a(\theta)\sigma^2(t,a)\frac{\partial}{\partial x}g_t^{L\theta}(x,\theta) - \frac{\partial}{\partial\theta}\big[g_t^{L\theta}(x,\theta)\mu(t,\theta)\big]\\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\big[g_t^{L\theta}(x,\theta)\sigma^2(t,\theta)\big],\\
\mathcal{N}^* g_t^{L\theta}(x,\theta) &= -\frac{\partial}{\partial\theta}\big[g_t^{L\theta}(x,\theta)\rho(t,\theta,Y_t)\sigma(t,\theta)\big].
\end{aligned}$$

I.e., $g_t^L(x)$ and $g_t^{L\theta}(x,\theta)$ are respectively the density and joint density of the local time L_t^x and θ together with L_t^x, in the formulas above the derivatives are consider in distributional sense, $\delta_a(\cdot)$ denotes the delta Dirac measure supported at $a \in \mathbb{R}$. The result than we want to state is that under certain conditions the functions $g_t^{L\theta}(x,\theta)$ and $g_t^L(x)$ satisfy the following *stochastic partial differential* equations, SPDEs, where the derivatives should be consider in the distributional sense.

$$\begin{aligned}
g_t^{L\theta}(x,\theta) &= g^{L\theta}(x,\theta) + \int_0^t \mathcal{L}^* g_s^{L\theta}(x,\theta)ds\\
&\quad + \int_0^t \big\{\mathcal{N}^* g_s^{L\theta}(x,\theta) + g_s^{L\theta}(x,\theta)(\alpha(s,\theta,Y_s) - \pi_s(\alpha))\big\}dB_s,\\
g_t^L(x) &= g_0^L(x) - \int_{0+}^t \sigma^2(s,a)\frac{\partial}{\partial x}g_s^{L\theta}(x,a)ds\\
&\quad + \int_{0+}^t\int_{-\infty}^{\infty} g_s^{L\theta}(x,\theta)(\alpha(s,\theta,Y_s) - \pi_s(\alpha))d\theta dB_s.
\end{aligned}$$

So the solutions of these SPDEs provide the relevant probabilities to solve the problem of risk default in more general situations. Imaging then the numerical challenges settled by these equations.

Optimal Stopping under Ambiguity.

FRANK RIEDEL, (University of Bunn).

14 December, 2007.
Consider the classical problem of optimal stopping in finance, that of the price of an American put option. An American option is a contract that once bought at time t_0 can be exercised at any later time before the maturity time T and that makes the profit $(K - S_s)^+$ at the right instant $s \in [t_0, T]$ when it is exercised, here as usual K is the fixed strike price at the time maturity T. Since this contract can be exercised at any time $s \in [t_0, T]$ intuitively one tries to find the time where exercising gives the best return, the price of a such American put option is found then to be,

$$U = \sup_{\tau} \mathbb{E}[e^{-r(T-t_0)}(K - S_\tau)|\mathcal{F}_{t_0}]. \tag{12}$$

In the classical theory it is assumed some fixed stochastic model for S_t the price of the underlying stock, a very interesting situation which is the central topic of this seminar is what could be done in the case where there is some uncertainty concerning the model for S_t and some responses are given in the discrete setting, then the title of the seminar becomes clear.

Statement of the problem
Let $(\Omega, \mathcal{F}, P_0, (\mathcal{F}_t))_{t\in\mathbb{N}}$ be a filtered probability space. Let $(X_t)_{t\in\mathbb{N}}$ be an adapted process, assumed bounded, that describes the payoff from stopping. The decision maker chooses a $\mathcal{F}_t$-stopping time τ with values in $\mathbb{N} \cup \{\infty\}$. From stopping she obtains a payoff $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$, she aims to maximize the expected reward and as she is uncertain about the distribution of X, she uses a class $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$. The (minimax) expected reward is thus given by

$$\inf_{P\in\mathcal{Q}} \mathbb{E}^P[X_\tau]. \tag{13}$$

I.e., she aims to reach the best possible reward when the conditions are the most adverse. We recall the main target said the maximization of $\inf_{P\in\mathcal{Q}} \mathbb{E}^P[X_\tau]$ over all stopping times $\tau \le T$. An essential assumption that lets develop the theory is the said, stability under pasting, or time-consistency of the family of probability measures Q.

Assumption 1 *The set of priors $\mathcal{Q}$ is time-consistent in the following sense. for P and Q in $\mathcal{Q}$, let (p_t) and (q_t) the density processes of P resp. Q with* *respect to a fixed reference measure P_0 which is also in $\mathcal{Q}$, i.e.*

$$p_t = \frac{dP}{dP_0}\Big|_{\mathcal{F}_t},$$

and analogously for Q. Fix some stopping time τ. Define a new probability measure R by setting for all $t \in \mathbb{N}$

$$\frac{dR}{dP_0}\Big|_{\mathcal{F}_t} = \begin{cases} p_t & \text{if} \quad t \leq \tau \\ \frac{p_\tau q_t}{q_\tau} & \text{else} \end{cases} \tag{14}$$

Then R belongs to $\mathcal{Q}$ as well.

Finally we will limit ourselves to enunciate the main results that carry out the solution of the problem, it is worth noting that they are generalizations of the basic martingale theory. We start by the central definition in this context

Definition 2 *Let $\mathcal{Q}$ be a set of priors. Let $(M_t)_{t\in\mathbb{N}}$ be an adapted process with $\mathbb{E}^p|M_t| < \infty$ for all $P \in \mathcal{Q}$ and $t \in \mathbb{N}$. (M_t) is called a* minimax (sub-super) martingale *with respect to $\mathcal{Q}$ if we have for $t \in \mathbb{N}$*

$$\operatorname{ess}\inf_{P\in\mathcal{Q}} \mathbb{E}^P[M_{t+1}|\mathcal{F}_t] = (\geq, \leq) M_t.$$

Lemma 1 *Let (M_t) be a bounded, adapted process.*

1. *M is a minimax submartingale if and only if it is a P-submartingale for all $P \in \mathcal{Q}$,*
2. *M is a minimax supermartingale if and only there exists $P^* \in \mathcal{Q}$ such that M is a P^*-supermartingale,*
3. *M is a minimax martingale with respect to $\mathcal{Q}$ if and only if*
   - *(a) there exists $P^* \in \mathcal{Q}$ such that M is a P^*-martingale and*
   - *(b) M is P-submartingale for all $P \in \mathcal{Q}$.*

Theorem 3 *Define the* minimal Snell envelope *of X with respect to $\mathcal{Q}$ recursively by $U_T = X_T$ and*

$$U_t = \max\left\{X_t, \operatorname{ess}\inf_{P\in\mathcal{Q}} \mathbb{E}^P[U_{t+1}|\mathcal{F}_t]\right\} \qquad (t = 0, \ldots, T-1). \tag{15}$$

Then

1. *U is the smallest minimax supermartingale with respect to $\mathcal{Q}$ that dominates X.*

2. *U is the value process of the optimal stopping problem under ambiguity, i.e.*

$$U_t = \operatorname{ess\,sup}_{\tau \geq t} \operatorname{ess\,inf}_{P \in \mathcal{Q}} \mathbb{E}^P[X_\tau | \mathcal{F}_t] \tag{16}$$

   The best reward when the conditions are the most adverse.

3. *an optimal rule is given by*

$$\tau^* = \inf\{t \geq 0 : U_t = X_t\}.$$

RESEARCH

During this time the question of adapting the method of optimal quantization which I was known in one of the first seminars, to a problem that concers doctoral thesis and consist on the numerical calculation of the following high dimensional integral

$$\iint_t^T \frac{\partial U_{\mathrm{BS}}}{\partial S}(\xi_T) e^{-\theta_s}(\eta_s - \eta_T) h(Y_s) \Psi(\xi_T, \eta_T, T; \theta_s, \eta_s, Y_s, s; y, t) d(\theta_s, \eta_s, Y_s, s, \eta_T, \xi_T).$$

Here $\Psi(\xi_T, \eta_T, T; \theta_s, \eta_s, Y_s, s; y, t)$ denots the joint density of involved processes(below), in the indicated times.

$$\begin{aligned}
\eta_s &:= 2 \int_t^s \sigma(Y_r)\sigma'(Y_r) e^{\theta_r} dr, \\
\theta_s &:= \int_t^s f'(Y_r) dr, \\
\xi_T &= \int_t^T \sigma^2(Y_s) ds, \\
f(y) &= \alpha(m - y) - \beta\gamma(y).
\end{aligned}$$

Here σ and γ are real functions with appropiate characteristics. I did not know the quatization algorithm befor, it seens that it adaptates better than Monte Carlo method in some situations, I belive then have in hands a very interesting non trivial application of it.